Nonabelian gauge theories on noncommutative spaces

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based on joint work with:

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Jurčo, Schupp, Wess, Nucl. Phys. **B604**, 148 (2001) [hep-th/0102129].
Jurčo, Schupp, Wess, "Noncommutative line bundle and Morita equivalence," hep-th/0106110.

Noncommutative Space

basic idea:

manifold M

$\longrightarrow$ algebra of functions on M: $C^\infty(M)$

$\longrightarrow$ noncommutative associative algebra $(\mathcal{A}_x, \star)$

fields ψ, A^i, F^{ij}, ... become "functions on a noncommutative space"

e.g. canonical structure:

$$[x^i \overset{\star}{,} x^j] = i\theta^{ij}, \quad \theta^{ij} \in \mathbb{C}$$

"coordinates" x^i generate $\mathcal{A}_x$

Weyl-Moyal star product

$$(f \star g)(x) = \exp\left(\frac{i\hbar}{2}\theta^{ij}\partial_i\partial'_j\right) f(x)g(x')|_{x=x'}$$

Gauge theory on a noncommutative space

can be based on a few basic ideas:

- covariant functions/coordinates
- locality (local star producs, NC fields depend on local gauge potential)
- gauge equivalence and consistency conditions

Gauge transformation of a field on a NC space:

$$\hat{\delta}\Psi = i\Lambda \star \Psi, \qquad \Psi, \Lambda \in \mathcal{A}_x$$

$$\text{note:} \quad \hat{\delta}(f \star \Psi) = if \star \Lambda \star \Psi \neq i\Lambda \star (f \star \Psi)$$

$\Rightarrow$ introduce covariant functions $\mathcal{D}f = f + \mathcal{A}(f)$ and in particular covariant coordinates $X^i \equiv \mathcal{D}x^i = x^i + A^i$, s.t.

$$\hat{\delta}X^i = i[\Lambda \overset{\star}{,} X^i] \quad \Leftrightarrow \quad \hat{\delta}A^i = i[\Lambda \overset{\star}{,} x^i] + i[\Lambda \overset{\star}{,} A^i]$$

Madore, Schraml, Schupp, Wess, Eur. Phys. J. **C16**, 161 (2000)

Noncommutative structure given by a $\star$-product, i.e.

$$f \star g = f \cdot g + \frac{i\hbar}{2}\theta^{ij}\partial_i f \cdot \partial_j g + \left(\frac{i\hbar}{2}\right)^2 \underbrace{B_2(f,g)}_{\text{bilinear}} + \dots$$

$\Rightarrow$ a classical limit exists

relation noncommutative fields $\longleftrightarrow$ classical fields?

$\Rightarrow$ SW maps $A[a]$, $\Lambda_\lambda[a]$, $\Psi[\psi, a]$, gauge equivalence conditions

$$\delta_\lambda A[a] = \widehat{\delta}_{\Lambda_\lambda[a]} A[a], \quad \delta_\lambda \Psi[\psi, a] = i\Lambda_\lambda[a] \star \Psi[\psi, a]$$

and consistency relation (from $[\delta_\alpha, \delta_\beta] = \delta_{-i[\alpha,\beta]}$)

$$[\Lambda_\alpha[a] \stackrel{\star}{,} \Lambda_\beta[a]] + i\delta_\alpha \Lambda_\beta[a] - i\delta_\beta \Lambda_\alpha[a] = \Lambda_{[\alpha,\beta]}[a].$$

Nonabelian NCGT via Seiberg-Witten map

The commutator of two Lie algebra-valued gauge parameters

$$[\Lambda \stackrel{\star}{,} \Lambda'] = \frac{1}{2}\{\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)\}[T^a, T^b] + \frac{1}{2}[\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)]\{T^a, T^b\}$$

is in general enveloping algebra-valued. (Exeption: U(n) in fundamental)

$\Rightarrow$ a priori ∞-many parameters! Luckily these can all be expressed in terms of classical gauge fields and parameters:

$$\Lambda_\alpha[a] = \alpha + \frac{1}{4}\theta^{ij}\{a_j, \partial_i\alpha\} + c_1\theta^{ij}\{\theta^{kl}(\partial_k - ia_k)f_{lj}, \partial_i\alpha\} + \ldots$$

$$\Psi[\psi, a] = \psi + \frac{1}{2}\theta^{ij}a_j\partial_i\psi + \frac{i}{4}\theta^{ij}a_ia_j\psi + \ldots$$

$$A_i[a] = a_i + \frac{1}{4}\theta^{kl}\{a_l, \partial_k a_i + f_{ki}\} + \ldots, \qquad F_{ij}[a] = \partial_i A_j - \partial_j A_i - i[A_i \stackrel{\star}{,} A_j]$$

Via the SW maps nonabelian NCGT is an ordinary gauge theory on a noncommutative space – with arbitrary gauge group.

Jurčo, Schraml, Schupp, Wess, Eur. Phys. J. **C17**, 521 (2000)

For constant θ the ordinary integral is a trace for the $\star$-product

$$\int f \star g = \int g \star f = \int f g.$$

An invariant action for the gauge potential and the matter fields is

$$S = -\frac{1}{4}\mathrm{tr} \int F_{ij} \star F^{ij} + \int \overline{\Psi} \star (\gamma^i D_i - m)\Psi, \qquad D_i\Psi \equiv \partial_i \Psi - i A_i \star \Psi.$$

Further reasons to consider the θ-expanded formulation:

- Perturbative expansion for small noncommutativity
- Solves IR problems that stem from IR/UV mixing (Vienna group)
- Important for nontrivial gauge fields (NC vector bundles)

Noncommutativity in String Theory

(is intrinsic through algebra of vertex operators)

Open strings in background B-field:

in decoupling limit at boundary points: $\langle x^i(\tau)x^j(\tau')\rangle_x = x^i \star x^j$

$\rightarrow$ string endpoints become noncommutative

($\star$ is the Weyl-Moyal star product for $\theta = B^{-1}$)

Let $B \rightarrow B + dA(x)$

gauge invariance: $\begin{cases} \text{Pauli-Villars} & \delta A_i = \partial_i \Lambda \\ \text{point-splitting} & \hat{\delta} A_i = \partial_i \Lambda + i[\Lambda \stackrel{\star}{,} A_i] \end{cases}$

require regularization-independence

$\rightarrow$ field-redefinitions $A[a]$, $\Lambda_\lambda[a]$ (Seiberg-Witten map)

with gauge-equivalence condition $\hat{\delta}_{\Lambda_\lambda[a]} A[a] = \delta_\lambda A[a]$

Seiberg, Witten, JHEP09, 32 (1999)

Construction of the Seiberg-Witten map

Abelian case:

• solution known explicitly for any Poisson structure $\theta(x)$ and corresponding Kontsevich $\star$-product. (Based on equivalent star-products)

Jurco, Schupp, Eur. Phys. J. **C14**, 367 (2000)

Jurčo, Schupp, Wess, Nucl. Phys. **B584**, 784 (2001)

• other approaches use NC Wilson lines (Ishibashi, Okuyama); recently the inverse SW map has been found (Wise & Mehen, Liu, Ooguri et al.).

Nonabelian case:

• can be reduced to abelian case via NC extra dimensions

(to order $\Theta^{\mu\nu}$: "Mini SW map": abelian $\rightarrow$ nonabelian GT)

• cohomological approach based on consistency relation

Finite gauge transformations, vector bundle and Morita equivalence

$\Lambda_\lambda[a]$ can be "exponentiated" $\rightarrow$ finite NC g.t. $G_g[a]$ for $g = \exp(i\lambda)$,

gauge equivalence and consistency relations: $(a_g = gag^{-1} + ig\mathrm{d}g^{-1})$

$$\mathcal{D}_{[a_g]}(f) = G_g[a] \star \mathcal{D}_{[a]}(f) \star G_g[a]^{-1}$$

$$\boxed{G_{g_1}[a_{g_2}] \star G_{g_2}[a] = G_{g_1 \cdot g_2}[a]}$$

This "NC group law" can be used to construct NC vector bundles.

The covariant coordinates $X^i \equiv x^i + A^i$ generate a new algebra $\star'$ via $\mathcal{D}(f \star' g) = \mathcal{D}f \star \mathcal{D}g$. We can show that $\star'$ and $\star$ are Morita equivalent; the equivalence module is a NC line bundle.

$\rightarrow$ back-reaction of NC gauge theory on NC space

Summary

covariant functions/coordinates – natural approach to gauge theory on noncommutative spaces

Seiberg-Witten map from ordinary to NC gauge theory – has been explicitly constructed for any Poisson structure $\theta(x)$

nonabelian NCGT – for any gauge group via SW-like maps (approach is also relevant for renormalizability)

global aspects, nontrivial gauge fields – finite NC gauge transformations, gauge equivalence and consistency relations;
NC vector bundles, Morita equivalence of $\star$-products